International Journal of Management, IT & Engineering
Vol. 8 Issue 12, December 2018,
ISSN: 2249-0558
Impact Factor: 7.119Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gate as well as in Cabell's Directories of Publishing Opportunities, U.S.A

On Some New Generalization Of Extended Beta Function

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ABSTRACT

In this paper, we introduce a new generalization of extended Beta function in terms of integral whose kernel contain the Mittag-Leffler function. We shall establish integral representation, differentiation formula, Beta distribution formula and various properties of new extended beta function. A number of new and known results are obtained as special cases.

Key Words : Beta function , Extended beta function, Mittag-Leffler function, Beta distribution .

1.INTRODUCTION

Special Functions have various applications in the field of Engineering, Mathematics and Statistical sciences. The classical Beta function given by Euler's is one of the essential special function.Eulers Beta function and its extentions have proved useful and necessary tool for the scientists and researchers. In recent years , several authors [3,4,5,9,11] have introduced a number of interesting and useful results related to extended beta function.

We start by recalling the classical Beta function B(α,β) defined by Euler [1,6,10] as

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt, \quad \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0 \tag{1}$$

In 1997, Chaudhary et al.[2] introduced the extension of Euler's beta function in the following form:

$$B(\alpha,\beta;p) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\left[-\frac{p}{t(1-t)}\right]dt, \tag{2}$$

(Re(p)>0; p=0, Re(α)>0, Re(β)>0)

The classical Mittag -Leffler function [7, 8] denoted by $E_\mu(.)$ is defined as

$$E_\mu(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\mu n+1)}$$

Where $\mu\ R_0^+$, $x \in \mathbb{C}$.

In this paper, we present a new generalization of extended beta function in the form of integral with Mittag -Leffler function as kernel, denoted by B(α, β ; p; σ ;μ) in the following way :

$$B(\alpha, \beta ; p; \sigma ;\mu) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} E_\mu\left[-\frac{p}{t^\sigma(1-t)^\sigma}\right] dt \quad (3)$$

(Re(p)>0; p=0, Re(α)>0, Re(β)>0 ;σ> 0; $\mu \epsilon R_0^+$)

Also, it preserves the symmetry of α , β parameter in the integrand .On replacing t by 1-t in (3), we have

B(α ,β ;p ;σ ;μ) = B(β ,α ;p ;σ ;μ)

For σ = 1, μ=1 in (3), we get definition (2) by Chaudhary et al.[2]. Further on putting σ =1, μ=1 and p =0 yields the Euler's beta function given by (1).

For σ = 1, we get the extension of beta function given by Shadab et al.[11] as follows

$$B(\alpha ,\beta ;p ;1 ;\mu) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} E_\mu\left[-\frac{p}{t(1-t)}\right] dt \quad (4)$$

(Re(α) >0 , Re(β) >0, Re (p) ≥ 0, μ ϵR_0^+)

Putting μ = 1 , (3) corresponds to the extended function by Lee et al. [5] as

$$B(\alpha, \beta ; p; \sigma ;1) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} E_\mu\left[-\frac{p}{t^\sigma(1-t)^\sigma}\right] dt \quad (5)$$

(Re(α) >0 , Re(β) >0, Re (p) ≥ 0, σ>0; μϵR_0^+)

2. INTEGRAL REPRESENTATIONS

In this section, the various integral representations of a new generalization of extended beta function B(α, β; p; σ ;μ) are derived.

Theorem 2.1. For B(α,β; p; σ ;μ) , the following integral representations hold:

$$B(\alpha,\beta;p;\sigma;\mu) = 2\int_0^{\pi/2} cos^{2\alpha-1}\theta sin^{2\beta-1}\theta\, E_\mu[sec^{2q}\theta cosec^{2q}\theta]\, d\theta \quad (6)$$

$$B(\alpha,\beta;p;\sigma;\mu) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} E_\mu[-p\,u^{-\sigma}(1+u)^{2\sigma}]du \ , \quad (7)$$

$$B(\alpha,\beta;p;\sigma;\mu) = 2^{1-\alpha-\beta} \int_{-1}^1 (1+u)^{\alpha-1} (1-u)^{\beta-1} E_\mu\left[\frac{-4^\sigma\ p}{(1-u^2)^\sigma}\right] du \ , \quad (8)$$

$$B(\alpha,\beta;p;\sigma;\mu) = (c-a)^{1-\alpha-\beta} \int_a^c (u-a)^{\alpha-1} (c-u)^{\beta-1} E_\mu\left[\frac{-p(c-a)^{2\sigma}}{(u-a)^\sigma (c-u)^\sigma}\right] du \quad (9)$$

(Re(p)>0; p=0, Re(α)>0, Re(β)>0 ;σ> 0; $\mu \epsilon R_0^+$)

Proof. By using the transformations t = $cos^2\theta$, $t = \frac{u}{1+u}$, $t = \frac{1+u}{2}$ and $t = \frac{u-a}{c-a}$ respectively in (3), we yield the required results.

Note : If $\mu = 1$,$\sigma = 1$then the results given in Chaudhary et al.[2 ,p22] are obtained as special cases.

Putting $\mu = 1$, we get the results by Lee et al.[5, p192].

For $\sigma = 1$, eqs (6) to (8) reduces to that by Shadab et al.[11, p239].

When p = 0 , we yield the corresponding results for classical beta function.

3. SOME PROPERTIES

In this section, we establish some properties of a new generalization of extended beta function B$(\alpha, \beta; p; \sigma; \mu)$.

Theorem 3.1. For B$(\alpha, \beta; p; \sigma; \mu)$, we have the following relation:

$$B(\alpha, \beta + 1; p; \sigma; \mu) + B(\alpha + 1, \beta; p; \sigma; \mu) = B(\alpha, \beta; p; \sigma; \mu) \quad (10)$$

Proof: Using (3) in L.H.S. of (10), we get

$$B(\alpha, \beta + 1; p; \sigma; \mu) + B(\alpha + 1, \beta; p; \sigma; \mu)$$

$$= \int_0^1 \{t^{\alpha-1}(1-t)^{\beta} + t^{\alpha}(1-t)^{\beta-1}\} E_\mu \left[-\frac{p}{t^\sigma(1-t)^\sigma}\right] dt$$

$$= \int_0^1 t^{(\alpha-1)}(1-t)^{(\beta-1)} \{(1-t) + t\} E_\mu \left[\frac{-p}{t^\sigma(1-t)^\sigma}\right] dt$$

$$= \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} E_\mu \left[\frac{-p}{t^\sigma(1-t)^\sigma}\right] dt$$

We get the R.H.S. of (10) .

Note: Taking μ =1 in eq(10), we get the relation given in [5]. Putting $\sigma = \mu = 1$, eq(10) gives the relation given in [2] and when p=0, we get the known results for classical beta function.

Theorem 3.2. For Re $(\alpha) > 0$,$Re(\beta) < 1$, Re (p) > 0; $\sigma, \mu \epsilon R_0^{+}$,n∈N

$$B(\alpha, 1 - \beta; p; \sigma; \mu) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} B(\alpha + n, 1; p; \sigma; \mu) \quad (11)$$

Proof : We know that

$$(1-t)^{-x} = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}, \ |t| < 1 \quad (12)$$

where $(x)_n = x(x+1)(x+2)............(x+n-1)$, n ϵN ; $(x)_0 = 1$.

On using (3) and (12) in the left hand side of (11) , we get

$$B(\alpha, 1 - \beta; p; \sigma; \mu) = \int_0^1 \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} t^{\alpha+n-1} E_\mu \left[\frac{-p}{t^\sigma(1-t)^\sigma}\right] dt$$

After interchanging the order of integration and summation and then using (3) ,we yield the desired result.

Theorem 3.3. For Re $(\alpha) > 0, Re(\beta) > 0, Re(p) > 0; \sigma, \mu \epsilon R_0^{+}$, n∈N ,

$$B(\alpha,\beta;p;\sigma;\mu) = \sum_{n=0}^{\infty} B(\alpha+n,\ \beta+1;\ p;\sigma;\mu) \tag{13}$$

Proof: Using (3) in L.H.S. of eq(13), we have

$$B(\alpha,\beta;p;\sigma;\mu) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} E_\mu\left[\frac{-p}{t^\sigma(1-t)^\sigma}\right] dt \tag{14}$$

Now, $(1-y)^{\beta-1}$ can be written in series expansion as follows

$$(1-y)^{\beta-1} = (1-t)^\beta \sum_{n=0}^{\infty} t^n, |t| < 1 \tag{15}$$

Using (15) in (14) and interchanging the order of integration and summation and then on making use of (3) we get the required result.

Note : Putting σ =1 in eq(11) and (13),we obtain the relation given in [11].

For μ =1, we get the results given in [5].

For $\sigma = \mu$ =1 , eq(11) and (13) reduces to the corresponding results in [2] .

4. BETA DISTRIBUTION

We know that Beta function and its various extensions has wide applications in the field of Statistics. Keeping that in view we define Beta distribution of the new generalization of extended beta function $B(\alpha,\beta;p;\sigma;\mu)$ as follows:

$$f(t) = \frac{1}{B(\alpha,\beta;p;\sigma;\mu)} t^{\alpha-1}(1-t)^{\beta-1} E_\mu\left[\frac{-p}{t^\sigma(1-t)^\sigma}\right], 0 < t < 1 \tag{16}$$

$f(t) = 0, otherwise.$

Let ν be any real number, then νth moment of X is given by

$$E(X^\nu) = \frac{B(\alpha+\nu,\beta;p;\sigma;\mu)}{B(\alpha,\beta;p;\sigma;\mu)} \tag{17}$$

($-\infty<\alpha<\infty$, $-\infty<\beta<\infty$; $p,\sigma,\mu>0$)

For $\nu = 1, we\ get$ the mean of the distribution as

$$E(X) = \frac{B(\alpha+1,\beta;p;\sigma;\mu)}{B(\alpha,\beta;p;\sigma;\mu)} \tag{18}$$

Variance of the distribution is defined by

$$E(X^2) - \{E(X)\}^2 = \frac{B(\alpha,\beta;p;\sigma;\mu)B(\alpha+2,\beta;p;\sigma;\mu) - [B(\alpha+1,\beta;p;\sigma;\mu)]^2}{[B(\alpha,\beta;p;\sigma;\mu)]^2} \tag{19}$$

The moment generating function of the distribution is

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n)$$

$$= \frac{1}{B(\alpha,\beta;p;\sigma;\mu)} \sum_{n=0}^{\infty} B(\alpha+n,\beta;p;\sigma;\mu) \frac{t^n}{n!} \tag{20}$$

The cumulative distribution is as follows

$$F(X) = \frac{B_X(\alpha,\beta;p;\sigma;\mu)}{B(\alpha,\beta;p;\sigma;\mu)} \tag{21}$$

Where $B_X(\alpha,\beta;p;\sigma;\mu)$ is the extended incomplete beta function which is defined as

$$B_X(\alpha,\beta;p;\sigma;\mu) = \int_0^X t^{\alpha-1}(1-t)^{\beta-1} E_\mu\left[\frac{-p}{t^\sigma(1-t)^\sigma}\right] dt$$

($-\infty<\alpha<\infty$, $-\infty<\beta<\infty$; $p,\sigma,\mu>0$)

For $p=0$, $\alpha>0$, $\beta>0$.

Note : By taking $\sigma=1$ and $\mu=1$ respectively, the above results corresponds to the definitions given in [11] and [5].

Further for $p=0$, $\sigma=\mu=1$, we get the results for classical beta function.

5. CONCLUSION

In this paper, we have introduced a new generalization of extended beta function using Mittag-Leffler function. We established its various properties such as integral representation, summation and beta distribution formulas. The various results derived in this paper are corresponding generalizations of known classical beta function and its extensions.

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